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Higher loops, integrability and the near BMN limit

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Abstract: In this note we consider higher-loop contributions to the planar dilatation operator of $\mathcal{N}=4$ SYM in the $\mathfrak{su}(2)$ subsector of two complex scalar fields. We investigate the constraints on the form of this object due to interactions of two excitations in the BMN limit. We then consider two scenarios to uniquely fix some higher-loop contributions: (i) Higher-loop integrability fixes the dilatation generator up to at least four-loops. Among other results, this allows to conjecture an all-loop expression for the energy in the near BMN limit. (ii) The near plane-wave limit of string theory and the BMN correspondence fix the dilatation generator up to three-loops. We comment on the difference between both scenarios.

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The complexity of perturbative calculations in quantum field theories increases dramatically with the loop order. While one-loop and perhaps two-loop Feynman diagrams can be worked out by hand, complete two-loop and three-loop calculations are accessible by means of computer algebra systems. Soon enough higher-loop investigations exceed the capacities of today's and tomorrow's computers. An immediate question is of course, *Do we need higher-loops?* In some sense it does not really matter whether we know n-loops or $(n+1)$-loops. However, the first few terms in a sequence sometimes enables one to guess how it continues. Knowing, e.g., the first four numbers of the sequence instead of three, reasonably increases the chances of a good guess. In some cases it might be possible to recast the sequence as a function of the coupling constant. Neglecting other, non-perturbative effects, this function can then be investigated in the strong coupling regime and might provide a test of the (weak/strong) AdS/CFT correspondence.

In this note we follow the lines of [1] and investigate the planar dilatation generator of $\mathcal{N}=4$ Super Yang-Mills theory in the $\mathfrak{su}(2)$ subsector consisting of two charged scalars Z, ϕ. The philosophy of that work was to write down the most general operator allowed by the Feynman diagrams that give rise to anomalous dimensions. The coefficients were then determined by fitting to known data. In the notation of that paper the two-loop dilatation operator was found to be

$$\begin{aligned}
D(g) &= \sum_{\ell=0}^{\infty}\left(\frac{g_{\mathrm{YM}}^2 N}{16\pi^2}\right)^{\ell} D_{2\ell}\,,\\
D_0 &= \{\}\,,\\
D_2 &= 2\{\} - 2\{0\}\,,\\
D_4 &= -8\{\} + 12\{0\} - 2\left(\{0,1\}+\{1,0\}\right)\,,\\
\{n_1, n_2, \ldots\} &= \sum_{k=1}^{L} P_{k+n_1,k+n_1+1} P_{k+n_2,k+n_2+1}\cdots
\end{aligned} \tag{1}$$

where $P_{k,k+1}$ interchanges the scalars at two adjacent sites.

To determine the unknown coefficients, the BMN limit was used as input. We consider a state $\operatorname{Tr} Z^J \phi^k$ + perm. with k excitations ϕ in a background of J background fields Z, where $k \ll J$. The BMN limit teaches us that the ℓ-loop anomalous dimension $\delta\Delta_{2\ell}$ should scale as [2]

$$\delta\Delta_{2\ell} \sim (\lambda')^{\ell}\left(1+\mathcal{O}\left(\frac{1}{J}\right)\right), \qquad \lambda' = \frac{g_{\mathrm{YM}}^2 N}{J^2}\,. \tag{2}$$

To be more precise, also the coefficients are known from plane wave string theory [2]

$$\Delta = J + \sum_{i=1}^{k}\sqrt{1+\lambda'\, n_k^2} + \mathcal{O}\left(\frac{1}{J}\right), \tag{3}$$

subject to the level matching constraint $\sum_{i=1}^{k} n_k = 0$. This all-loop conjecture was confirmed using gauge theory means in [3].

The limit (2) and the energy formula (3) constrain the most general dilatation operator. It is easily seen that in a dilute gas of excitations, the dilatation operator should act on the position of a single ϕ in a background of Z's as [4]

$$D_{2\ell} \approx -2C_\ell \,\Box^\ell\,, \tag{4}$$

where $\Box$ is the lattice laplacian. In the dilute gas approximation (4) gives

$$\delta\Delta_{2\ell} = (-1)^{i+1}\,2^{1-2\ell} C_\ell \sum_{i=1}^{k} \left(\lambda'\,n_k^2\right)^\ell \tag{5}$$

which matches exactly (3) when $C_{1,2,3,4,\ldots} = (1,1,2,5,\ldots)$ are the Catalan numbers governing the expansion of the square root. Using this constraint sufficed to determine all coefficients of the two-loop operator (1).

For the case of two excitations, $\operatorname{Tr} Z^J \phi^2 + \text{perm.}$, the one-loop dilatation operator can be diagonalised explicitly [5]. Once this is done, all higher-loop anomalous dimensions can be obtained exactly in perturbation theory. The all-loop generalisation of the three-loop energy formula presented in [1] appears to be[1]

$$\Delta_n^J := J + 2 + \sum_{\ell=1}^{\infty} \left(\frac{g_{\mathrm{YM}}^2 N}{\pi^2} \sin^2 \frac{\pi n}{J+3}\right)^\ell \left(c_\ell + \sum_{k,m=1}^{\ell-1} c_{\ell,k,m} \frac{\cos^{2m} \frac{\pi n}{J+3}}{(J+3)^k}\right) \tag{6}$$

where the first few coefficients derived from (1) are given by

$$c_1 = 1\,, \qquad c_2 = -\frac{1}{4}\,, \qquad c_{2,1,1} = -1\,. \tag{7}$$

This formula interpolates smoothly between the regime of large J all the way down to the Konishi operator. Reassuringly, (6) gives the correct two-loop scaling dimension Δ_0^1 of the Konishi operator!

In order to fix the three-loop contribution, the BMN limit turned out to be not sufficient. There are six free coefficients and (4) fixes only four of them. Further insights were needed to find the remaining coefficients. It was shown by Minahan and Zarembo that the planar dilatation generator in the subsector of scalars, including the $\mathfrak{su}(2)$ subsector under consideration, is integrable [6].[2] Interestingly, this integrability seems to extend (in a perturbative sense) also to the two-loop contribution (1),[1]. Therefore it is a reasonable assumption that also the three-loop contribution would exhibit integrability. This assumption was used to determine the remaining two coefficients of the three-loop dilatation operator. It was attempted to go to four-loops in this way, unfortunately, it seemed that a *single* coefficient out of twelve could not be determined.

However, this is not the case: As we will see, even the *four-loop* contribution to the dilatation generator is uniquely fixed using (a) the BMN limit and (b) integrability! The

[1]Note that the expression Δ_n^J gives the scaling dimensions of the superconformal primary states with two excitations. Here, $2 + \Delta_n^{J-2}$ gives the scaling dimension for the descendant state $\operatorname{Tr} Z^J \phi^2 + \text{perm.}$.

[2]Recently, this result was extended to the full one-loop planar $\mathcal{N} = 4$ SYM theory [7].

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constraint (4) governing the propagation of a single excitation is a *necessary* condition for the BMN limit, it is however, not *sufficient*. There are further constraints which were overlooked in [1]. Namely, also the interactions of two or more excitation must obey a very specific pattern in order not to spoil (2). This can be seen in the following way. For pairwise interactions the excitations must be close, due to phase space considerations these interactions are then suppressed by $1/J$. For the BMN-limit (2), however, they would need to be suppressed by $1/J^{2\ell+1}$. Indeed, for D_2, D_4 this happens to be the case. In contrast, for D_6 the most general contribution which obeys (4), gives $\delta\Delta_6 \sim 1/J^5$ in violation of (2). Only if the coefficients of D_6 are arranged in a very specific way[3] we will get $\delta\Delta_6 \sim 1/J^6$. This is fortunate, as we can impose novel constraints on the coefficients of the higher-loop contributions. As we shall see, it allows to fix at least one further coefficient of D_6 from the BMN limit alone. The same happens for the four-loop contribution; in order to have $\delta\Delta_8 \sim 1/J^8$ we can fix at least two additional coefficients.

Let us now reinvestigate the three-loop contribution D_6. We demand that the BMN limit (2) exists, i.e. $\delta\Delta_6 \sim 1/J^6$. Note that we will *not* require that the BMN energy formula (3) is reproduced correctly. Four coefficients are fixed as follows

$$\begin{aligned} D_6 = {} & (60 + 6\alpha_1 - 56\alpha_2)\{\} + (-104 + 14\alpha_1 + 96\alpha_2)\{0\} + \\ & +(24 + 2\alpha_1 - 24\alpha_2)\left(\{0,1\} + \{1,0\}\right) + (4 + 6\alpha_1)\{0,2\} + \\ & +(-4 + 4\alpha_2)\left(\{0,1,2\} + \{2,1,0\}\right) - \alpha_1\left(\{0,2,1\} + \{1,0,2\}\right). \end{aligned} \tag{8}$$

From this we can derive the following coefficients for the energy formula (6) of states with two excitations

$$c_3 = \frac{1}{8} - \frac{1}{8}\alpha_2\,, \qquad c_{3,k,m} = \begin{pmatrix} +\frac{3}{4} + \frac{3}{4}\alpha_1 & +\frac{1}{2} - \alpha_1 - 2\alpha_2 \\ -\frac{3}{4} & +\frac{5}{2} \end{pmatrix}. \tag{9}$$

At this point we need further input to fix the remaining coefficients. There are two ways in which to proceed: (i) We rely on higher-loop integrability or, (ii) we use the near plane-wave limit [8, 9].

In the case (i) we investigate degenerate pairs of operators [1]. For states of length $7, 8$ with 3 excitations there is one degenerate pair each. Three-loop degeneracy requires that

$$\alpha_1 = \alpha_2 = 0 \tag{10}$$

in agreement with the result of [1]. The other pairs are then also three-loop degenerate, which can be proved by showing that the second integrable charge extends to three-loops [1]. There are some interesting points regarding this solution. For one, integrability fixes exactly the right number of coefficients for a unique solution. Moreover, the constraints from integrability are compatible with the constraints from the BMN limit. Most importantly, we have only demanded the *qualitative* BMN limit (2). Integrability fixes the remaining coefficients in just the right way for the *quantitative* BMN limit (3), i.e. $c_3 = +1/8$.

[3]It would be interesting to find a general criterion which determines whether an interaction of two or more excitations respects the BMN limit or not.

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As the procedure seemed very successful at the three-loop level we now turn to four-loops. There we have 12 distinct structures and corresponding coefficients. Here, the existence of the BMN limit determines 6 coefficients, 4 from the propagation of a single excitation and 2 from pairwise interactions. Integrability determines 5 further coefficients. As explained in [1], the remaining coefficient β multiplies the structure $[[D_4, D_2], D_2]$; it corresponds to a rotation of the space of states generated by $g^6[D_4, D_2]$ and thus does not influence scaling dimensions. The resulting four-loop dilatation generator is

$$\begin{aligned}
D_8 = &-560\{\} + (1036 + 4\beta)\{0\} + \\
&+(-266 - 4\beta)(\{0,1\} + \{1,0\}) + (-66 - 2\beta)\{0,2\} - 4\{0,3\} + \\
&+4(\{0,1,3\} + \{0,2,3\} + \{0,3,2\} + \{1,0,3\}) + \\
&+(78 + 2\beta)(\{0,1,2\} + \{2,1,0\}) + (-18 + 2\beta)(\{0,2,1\} + \{1,0,2\}) + \\
&+(1 - \beta)(\{0,1,3,2\} + \{0,3,2,1\} + \{1,0,2,3\} + \{2,1,0,3\}) + \\
&+(6 - 2\beta)\{1,0,2,1\} + 2\beta(\{0,2,1,3\} + \{1,0,3,2\}) - \\
&-10(\{0,1,2,3\} + \{3,2,1,0\}) ,
\end{aligned} \tag{11}$$

it equals the structure given in [1] with $\alpha = 3$. The four-loop coefficients of the energy formula (6) are given by[4]

$$c_4 = -\frac{5}{64}, \quad c_{4,k,m} = \begin{pmatrix} -\frac{5}{8} & -\frac{5}{12} & -\frac{1}{3} \\ +\frac{3}{4} & -\frac{7}{4} & -\frac{7}{2} \\ -\frac{1}{2} & +\frac{59}{12} & -\frac{49}{6} \end{pmatrix}. \tag{13}$$

Again, we see that the BMN energy, $c_4 = -5/64$, is predicted correctly by integrability. Again, the BMN limit and integrability fix a complementary set of coefficients. It remains to be seen whether also the five-loop dilatation generator can be obtained in this way; it involves approximately sixty structures.

A further application of (6) is the near BMN limit of $\mathcal{O}(1/J)$ corrections to the energy. Some inspired guessing yields an all-loop expression for the near BMN limit which agrees at four-loops

$$\Delta_n^J = J + 2\sqrt{1 + \lambda' n^2} - \frac{8\lambda' n^2}{J\sqrt{1 + \lambda' n^2}} + \frac{2\lambda' n^2}{J(1 + \lambda' n^2)} + \mathcal{O}\left(\frac{1}{J^2}\right). \tag{14}$$

The first $1/J$ term can be regarded as a renormalisation of the coupling constant. If we replace J in the definition of λ' by $J + 4$ this term can be absorbed into the leading order energy.

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[4] Using these coefficients the energy of the Konishi operator appears to be

$$\Delta_{\mathcal{K}} = \Delta_0^1 = 2 + \frac{3g_{\mathrm{YM}}^2 N}{4\pi^2} - \frac{3g_{\mathrm{YM}}^4 N^2}{16\pi^4} + \frac{21 g_{\mathrm{YM}}^6 N^3}{256\pi^6} - \frac{705 g_{\mathrm{YM}}^8 N^4}{16384\pi^8} + \cdots .$$

However, in this case it is not clear whether (6) applies, because the length of the interaction D_8, 5, exceed the length of the state, 4. The above expression therefore represents an extrapolation of the range of validity of D_8, which might or might not be true.

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Unfortunately, the formula (14) does not agree with the expression for the near plane-wave limit derived in [9][5]

$$\Delta_n^J = J + 2\sqrt{1+\lambda' n^2} - \frac{4\lambda' n^2}{J\sqrt{1+\lambda' n^2}} - \frac{2\lambda' n^2}{J} + \mathcal{O}\left(\frac{1}{J^2}\right). \tag{15}$$

It agrees with (6) only up to two-loops [9]. In scenario (ii) we achieve agreement by matching the remaining coefficients α_1, α_2 at three-loops to (15) and get

$$\alpha_1 = 2, \quad \alpha_2 = 0. \tag{16}$$

This choice of coefficients lifts the degeneracy of pairs at the three-loop level and thus violates integrability. Furthermore, the input from the near plane-wave limit is not sufficient to determine four-loops unambiguously.

We conclude that higher-loop integrability contradicts with the result (15) of [9]. At this point it is not clear which information to use in order to construct the three-loop dilatation generator D_6. On the one hand, given the long list of verifications of the BMN correspondence, it is hard not to believe in scenario (ii). Nevertheless, the all-loop conjecture (14) of (i) is rather similar to the near plane-wave result (15) of (ii). Maybe a slight modification of the relations is required to match both expressions? E.g. one might imagine that the effective coupling constant λ' is defined using $J+2$ instead of J, which apparently makes one term agree. On the other hand, a failure of higher-loop integrability would be disappointing as well. Here, it was demonstrated that integrability and the existence of a BMN limit go hand in hand to determine the dilatation generator up to four-loops. What is more, they predict the leading order BMN energy correctly. This appears to be more than a coincidence and makes scenario (i) very attractive: Whether or not related to $\mathcal{N}=4$ SYM, this scenario requires further investigations. Eventually, only a derivation of the three-loop dilatation generator with as few assumption as possible may decide in either direction. As was demonstrated in [10] the superconformal algebra imposes some valuable constraints that might pave the way.

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